4.2 - Reduction of Order

Consider a 2^{nd}-order DE

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0 \qquad (1)$$

If y_1, y_2 form a fundamental set of solutions, then they are lin. indep.

So if $c_1 y_1 + c_2 y_2 = 0$ with $c_1, c_2 \neq 0$,

then we can write $y_2 = \boxed{-\frac{c_1}{c_2}} y_1$,

where $-\frac{c_1}{c_2}$ is a function of x.

Then $y_2 = \underbrace{u(x)}_{\text{unknown function of } x.} y_1$

In standard form, (1) becomes

$$y'' + P(x)y' + Q(x)y = 0$$

If y_1 is a known solution, let's

Consider $y_2 = u(x)\, y_1(x)$

$$y_2' = u'(x)\, y_1(x) + u(x)\, y_1'(x)$$

$$y_2'' = u''(x)\, y_1(x) + 2u'(x)\, y_1'(x) + u(x)\, y_1''(x)$$

substitute:

$$u'' y_1 + 2u' y_1' + u\, y_1'' + P(u' y_1 + u y_1') + Q\, u y_1 = 0$$

Note: $u(y_1'' + P y_1' + Q y_1) = 0$

we reduced the order

$$u''y_1 + 2u'y_1' + Pu'y_1 = 0$$

$$u''y_1 + (2y_1' + Py_1)\,u' = 0$$

Let $w = u'$. Then $w' = u''$

$$w'y_1 + (2y_1' + Py_1)\,w = 0 \qquad \text{(separable, linear)}$$

$$\int \frac{dw}{w} = -\int \left(\frac{2y_1(x)'}{y_1(x)} + P(x) \right) dx$$

$$\ln|w| = -\ln y_1^2 - \int P(x)\,dx$$

$$\ln|w y_1^2| = -\int P(x)\,dx$$

$$w y_1^2 = e^{-\int P(x)\,dx}$$

$$u' = w = \frac{e^{-\int P\,dx}}{y_1^2}$$

$$\boxed{u = \int \frac{e^{-\int P\,dx}}{y_1^2}\,dx}$$

Reduction of order formula

$$y_2 = u\,y_1 \ ; \quad y = c_1 y_1 + c_2 y_2$$

The indicated function $y_1(x)$ is a solution of the given differential equation. Use reduction of order (the process or the formula) to find a second solution $y_2(x)$.

Ex: $y'' + 9y = 0; \quad y_1 = \sin 3x$

$$y'' + P(x)y' + Q(x)y = 0$$

std form ✓

$$y_2 = u\,y_1, \text{ where } u = \boxed{\int \frac{\boxed{e^{-\int P\,dx}}}{y_1^2}\,dx}$$

$$P(x) = 0 \qquad e^{\int 0} = e^{c_1} = C \text{ (constant)}$$

$$u = \int \frac{c}{\sin^2 3x}\,dx = c\int \csc^2 3x\,dx$$

$$\frac{d}{dx}(-\cot x) = \csc^2 x$$

$$\tan x \qquad \sec^2 x$$

$$u = -\frac{c}{3}\cot 3x$$

$$y_2 = u\,y_1 = -\frac{c}{3}\cot 3x \sin 3x$$

$$= -\frac{c}{3}\cos 3x$$

$$\boxed{y_2 = \cos 3x}$$

$$y = c_1 y_1 + c_2 y_2$$

$$y = c_1 \sin 3x + c_2 \cos 3x$$

given   we found

Ex: $6y'' + y' - y = 0; \quad y_1 = e^{x/3}$

$y'' + \frac{1}{6}y' - \frac{1}{6}y = 0$ (Std form)

$P(x) = \frac{1}{6}$

Note: $\boxed{u = \int \frac{e^{-\int P dx}}{y_1^2} dx}$

$e^{-\int P dx} = e^{-\frac{1}{6}x}$

$u = \int \frac{e^{-\frac{1}{6}x}}{e^{2/3x}} dx$

$e^{(\frac{1}{6} + \frac{2}{3})x} = e^{\frac{5}{6}x}$

$u = \int e^{-\frac{5}{6}x} dx = -\frac{6}{5} e^{-\frac{5}{6}x} + C$

$y_2 = u y_1 = -\frac{6}{5} e^{-\frac{5}{6}x} e^{x/3} + C e^{x/3}$

we don't do this

$y = c_1 y_1 + c_2 y_2 = c_1 e^{x/3} + -\frac{6}{5} c_2 e^{-1/2 x} \cancel{+ c_2 C e^{x/3}}$

$\boxed{y_2 = e^{-x/2}}$

$y = c_1 e^{x/3} + c_2 e^{-x/2}$
(general solution)

Ex: $x^2y'' - 3xy' + 5y = 0; \quad y_1 = x^2\cos(\ln x)$

$$y'' - \frac{3}{x}y' + \frac{5}{x^2}y = 0 \qquad u = \int \frac{e^{-\int P dx}}{y_1^2}dx$$

$$P(x) = -\frac{3}{x} \Rightarrow e^{-\int P dx} = e^{\int \frac{3}{x}dx}$$

$$= e^{3\ln x} = e^{\ln x^3} = x^3$$

$$u = \int \frac{x^3}{x^4\cos^2(\ln x)}dx = \int \frac{1}{x}\sec^2(\ln x)\,dx$$

$$= \tan(\ln x)$$

$$y_2 = x^2\cos(\ln x)\tan(\ln x)$$

$$\boxed{y_2 = x^2\sin(\ln x)}$$

Ex: The indicated function $y_1(x)$ is a solution of the associated homogeneous equation. Use the method of reduction of order [the process] to find a second solution $y_2(x)$ of the homogeneous equation and a particular solution $y_p(x)$ of the given nonhomogeneous equation.

$y'' - 4y' + 3y = x; \quad y_1 = e^x$

Dr. Matt's recommendation: For nohomogeneous DEs, use the process, not the formula.

$$y_2 = uy_1 \Rightarrow y_2 = e^x u$$

$$y' = e^x u + e^x u', \quad y'' = e^x u + e^x u' + e^x u' + e^x u''$$

$$y'' = e^x u + 2e^x u' + e^x u''$$

$$y'' - 4y' + 3y = x$$

$$\cancel{e^x u} + 2e^x u' + e^x u'' - \cancel{4e^x u} - 4e^x u' + \cancel{3e^x u} = x$$

$$e^x u'' - 2e^x u' = x$$

$$w = u' \Rightarrow w' = u''$$

$$e^x w' - 2e^x w = x \qquad \text{(linear)}$$

$$w' - 2w = x e^{-x}$$

$$\mu = e^{-2x}$$

$$\int \frac{d}{dx}\left(e^{-2x} w\right) dx = \int x e^{-3x} dx$$

$$s = x \quad dt = e^{-3x} dx$$
$$ds = dx \quad t = -\tfrac{1}{3} e^{-3x}$$

$$e^{-2x} w = -\tfrac{1}{3} x e^{-3x} - \tfrac{1}{9} e^{-3x} + C$$

$$u' = w = -\tfrac{1}{3} x e^{-x} - \tfrac{1}{9} e^{-x} + C_1 e^{2x}$$

$$s = -\tfrac{1}{3} x \quad dt = e^{-x} dx$$
$$ds = -\tfrac{1}{3} dx \quad t = -e^{-x}$$

$$c = \frac{C_1}{2}$$

$$u = \tfrac{1}{3} x e^{-x} + \underline{\tfrac{1}{3} e^{-x}} + \underline{\tfrac{1}{9} e^{-x}} + C e^{2x}$$

$$y_2 = u y_1 = \underbrace{\tfrac{1}{3} x + \tfrac{4}{9}}_{y_p} + \underbrace{C e^{3x}}_{\text{gives us } y_2}$$

$$\boxed{y_2(x) = e^{3x}, \quad y_p(x) = \frac{1}{3}x + \frac{4}{9}}$$

Solution is

$$y = c_1 e^{x} + c_2 e^{3x} + \frac{1}{3}x + \frac{4}{9}$$